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COMPLEX VARIABLE METHODS FOR FATIGUE SENSITIVITY ANALYSIS (PREPRINT)

A. Voorhees, R. Bagley, and H. Millwater

University of Texas at San Antonio

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14. ABSTRACT
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Complex Variable Methods for Fatigue Sensitivity Analysis

A. Voorhees[1], R. Bagley[2], H. Millwater[3],
[1]Graduate student, [2]Professor, [3]Associate Professor
Department of Mechanical Engineering, Univ. of Texas at San Antonio

Abstract

Complex variable, numerical differentiation techniques have proven useful in many fields of engineering analysis. Complex Taylor series expansion and Fourier differentiation are two such complex variable methods. This paper adapts the use of both complex Taylor series expansion and Fourier differentiation for fatigue sensitivity analysis. The sensitivity of the number of cycles to failure to input parameters and initial conditions has been determined by traditional numerical differentiation techniques as well as through complex variable methods. Both complex Taylor series expansion and Fourier differentiation have been found to be more accurate and stable than traditional central differencing.

Introduction

The fatigue of aircraft components due to repeated loading cycles is a costly problem that may also poses safety risks to pilots and crewmembers. If a part fails due to fracture it can mean costly repairs or even retirement of the plane. The design and maintenance of aircraft should take into account the effects of fatigue. The goal of proper design and maintenance should be to increase the life of the aircraft while ensuring its safety. At the same time it is desirable to reduce the cost of operation and maintenance. In order to determine the optimal design and maintenance procedures, it is necessary to know the relative importance of the variables in the problem. This relative importance can be measured by the sensitivity or the partial derivative. If the sensitivity information was known, it would become possible to run an optimization routine to determine the optimal dimensions, or material properties of a part. It would also be possible to determine the optimal parameters of an inspection routine designed to identify small cracks before they threaten the life of the part. Traditional numerical differentiation should allow for the calculation of such sensitivities, however newer complex variable methods promise more accuracy, especially in the calculation of higher order derivatives.

Numerical differentiation techniques such as finite differencing are commonly used to evaluate the derivatives of implicit functions. In the simplest form of finite differencing the derivative of a function at a point can be calculated by evaluating the function at a second nearby point located along the real axis. This nearby point is the sample point, and the distance between the sample point and the original point is the sampling radius. The difference in the value of the function evaluated at the sample point and at the original point divided by the sampling radius yields an estimate of the first derivative. Higher order derivatives as well as more theoretically accurate derivatives can be calculated by evaluating the function at more sample points and performing more difference operations. The finite difference method is easy to perform, however it is only accurate when the difference between the sample points is small. When subtracting small numbers, machine round off can also introduce error meaning that there is a limit to the accuracy of the method. This becomes more of a problem when calculating higher order derivatives, since more subtraction operations are required.

An alternative to finite differencing is complex Taylor series expansion (CTSE). CTSE was originally described by Lyness and Moler in the 1960's but wasn't brought to the attention of the engineering community until the 1990's when Squire showed the accuracy of the method for computing the first derivative of several explicit functions [1][2][3]. In CTSE, the sample point is complex. The first derivative is calculated simply by determining the imaginary component of the function evaluated at the sample point. For the first derivative CTSE involves no difference operation and the theoretical error is smaller than for finite differencing. In order to calculate the higher order derivatives, additional sample points need to be taken and more difference operations become necessary.

Fourier Differentiation (FD) is another alternative method using sampling in the complex plane. Fourier differentiation requires the evaluation of several sample points along a circular contour around the initial point in the complex plane. This technique was first described by Lyness et al in 1967 [1][2]. The function is evaluated at these sample points and a fast Fourier transform (FFT) of the evaluated sample points is used to calculate the derivative. The use of the FFT as a method to calculate the derivatives was described by Lyness et al [4]. More recently, FD was shown to be useful in the computation of senstivities of implicit functions [5]. One benefit of FD is that the results of the FFT also contain higher order derivatives. The number of derivatives that can be obtained from the FFT is related to the number of sample points chosen. Furthermore, FD is theoretically more accurate than CTSE and finite differencing.

CTSE has already caught on in many engineering fields. In fluid dynamics, CTSE has been used to find sensitivities for the solution of the Navier-Stokes equation [6]. Furthermore researchers have been able to apply CTSE techniques to finite element methods in the field of aerodynamics and aero-structural analysis [7][8]. CTSE has also proven to be a valuable technique in the study of heat transfer [9], dynamic system optimization [10], and has even been incorporated into pseudospectral [11] and eigenvalue sensitivity methods [12]. A thorough review of the literature has not turned up any instances of CTSE being applied to fatigue problems. Standard finite difference techniques have been used in fatigue problems [13][14], as well as probabilistic sensitivity methods [15]. By adopting complex variable numerical differentiation techniques, it should be possible to obtain more accurate sensitivity estimates.

The goal of this paper is to explore the use of complex variable methods to determine sensitivities of the cycles-to-failure with respect to input quantities such as loading amplitude, crack growth parameters, initial crack size, fracture toughness, etc.

Additionally higher order derivatives will be calculated which may be used to create Taylor series estimates of the number of cycles to failure as a function of the aforementioned initial conditions and properties.

II. Methodology

A. Numerical Differentiation

The derivative of a real function can be defined as the instantaneous change in the value of the function at a given location. The limit definition of the derivative states:

$$f'(x_o) = \lim_{x \to x_o} \frac{f(x) - f(x_o)}{x - x_o}. \tag{1}$$

where $f'(x_o)$ is the derivative of $f(x_o)$. The use of this definition requires $f(x)$ to be continuous at x_o. In situations where this limit can not be evaluated easily or exactly (i.e. when $f(x)$ is implicit) $f'(x_o)$ can be calculated by numerical approximation of the limit. The most traditional approximation method is finite differencing. The finite differencing method approximates the derivative by setting $f(x)$ in equation 1 to $f(x_o+h)$ where h is very small.

$$f'(x_o) \approx \frac{f(x_o + h) - f(x_o)}{h} \tag{2}$$

This approach requires the evaluation of the function at two points, the initial point x_o and the sample point x_o+h. When h is positive, the method is referred to as forward differencing and when h is negative it is referred to as backwards differencing. A more accurate approximation can be obtained by averaging the forward and backwards differences for equal values of h.

$$f'(x_o) \approx \frac{f(x_o) - f(x_o - h)}{2h} + \frac{f(x_o + h) - f(x_o)}{2h} = \frac{f(x_o + h) - f(x_o - h)}{2h} \tag{3}$$

This averaging technique is called central differencing, and is often the method of choice in numerical differentiation. The increased accuracy of central differencing arises from the cancellation of even, higher ordered terms in the Taylor series. This pushes the accuracy from $O(h)$ for forward differencing to $O(h^2)$ for central. In this paper, central differencing was selected for comparison with CTSE and FD. Figure 1.A displays the location of sampling points in the complex plane.

Finite differencing can also be used to evaluate higher order derivatives. The second derivative of $f(x)$ can be defined as the derivative of a derivative.

$$f^{(2)}(x_o) = \lim_{x_1 \to x_o} \frac{f'(x_1) - f'(x_o)}{x_1 - x_o} = \frac{\lim_{x \to x_o} \frac{f(x_2) - f(x_1)}{x_2 - x_1} - \lim_{x \to x_o} \frac{f(x_1) - f(x_o)}{x_1 - x_o}}{x_1 - x_o} \tag{4}$$

Approximating the limits by sampling $f(x)$ at $f(x_o)$, $f(x_o+h)$, and $f(x_o+2h)$ where h is small the second forward derivative is.

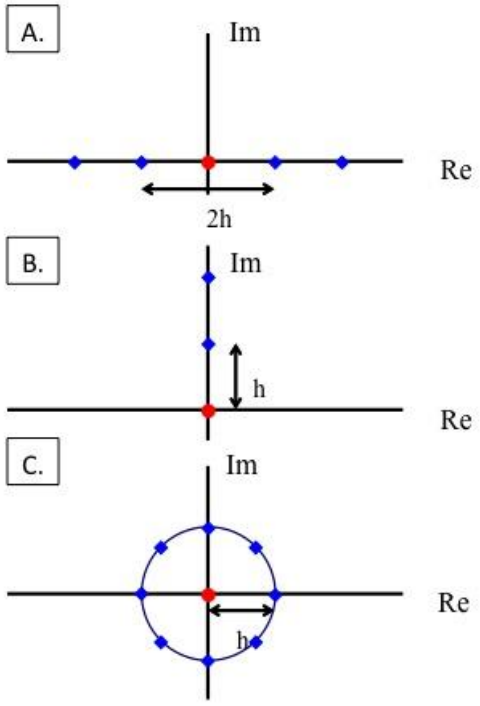

Figure 1. The Location of Sampling Point in the Complex Plane. A. The location of the first 4 sampling points centered around the initial point in central differencing. B. The location of the first two sample points in CTSE. C. 8 sample points for FD.

$$f^{(2)}(x_o) \approx \frac{\frac{f(x_o+2h)-f(x_o+h)}{h} - \frac{f(x_o+h)-f(x_o)}{h}}{h} \tag{5}$$

$$f^{(2)}(x_o) \approx \frac{f(x_o+2h)-2f(x_o+h)+f(x_o)}{h^2}$$

Note here that two sample points are needed to calculate the second derivative. The central difference for the second derivative is calculated by sampling *f(x)* at $f(x_o-h)$, $f(x_o)$, and $f(x_o+h)$.

$$f^{(2)}(x_o) \approx \frac{f(x_o+h)-2f(x_o)+f(x_o-h)}{h^2} \tag{6}$$

Finite differencing techniques can be extended to higher order derivatives still. Unfortunately this also requires additional sample points. When the number of sample points increases the error can become quite large. This error comes from two different sources. The first source is subtractive cancellations in which the difference between two sample points becomes too small to be accurately calculated. This type of error occurs mainly in higher order derivatives when the step size is too small. The second source of error comes from the approximation itself. This places the user at a crux, as h must be selected so as to be neither too small nor too big.

By sampling in the complex plane rather than solely along the real axis, some of these problems can be remedied. Derivatives can also be defined in the complex plane using limits

$$f'(z_o) = \lim_{z \to z_o} \frac{f(z)-f(z_o)}{z-z_o}, \tag{7}$$

where z is the complex variable. In order for a function to have a derivative at $z = z_o$ it must be analytic in the neighborhood surrounding z_o. A function is analytic if its derivative is identical no matter what direction in the complex plane z approaches z_o from.

If the complex function is defined as

$$f(z) = f(x + i \cdot y) = u(x,y) + i \cdot v(x,y), \tag{8}$$

then the derivative can be written as

$$f'(z) = \frac{\partial u(x,y)}{\partial x} + i \cdot \frac{\partial v(x,y)}{\partial x} = \frac{\partial u(x,y)}{\partial y} + i \cdot \frac{\partial v(x,y)}{\partial y}. \tag{9}$$

The Cauchy-Riemann equations state that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{10}$$

which allows equation 9 to be rewritten as

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial v}{\partial x}. \tag{11}$$

When the function of interest is real valued, only the real part of *f'(z)* need be considered.

$$real(f'(z)) = f'(x) = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}. \tag{12}$$

This reveals that the real part of the first derivative can be determined by examining the change along either the real axis (*x*) or the imaginary axis (*y*). The limit defined by the partial of u with respect to x is the same as the one approximated by finite differencing.

$$f'(x_o) = \frac{\partial u}{\partial x} = \lim_{x \to x_o} \frac{u(x,y_o) - u(x_o,y_o)}{x - x_o} \approx \frac{u(x,y_o) - u(x_o,y_o)}{x - x_o} \tag{13}$$

Approximating the limit defined by the partial of *v* with respect to *y* leads to a whole new type of numerical differentiation.

$$f'(x_o) = \frac{\partial v}{\partial y} = \lim_{y \to y_o} \frac{v(x_o,y) - v(x_o,y_o)}{y - y_o} \approx \frac{v(x_o,y) - v(x_o,y_o)}{y - y_o} \tag{14}$$

Replacing y with h, and accounting for the fact that y_o and $v(x_o,y_o)$ are zero due to the fact that the initial point is real valued, equation 14 becomes.

$$f'(x_o) \approx \frac{v(x_o,h)}{h} = \frac{imag(f(x_o + ih))}{h} \tag{15}$$

In this manner, the derivative can be approximated by adding a small imaginary component to x_o. this method is referred to as CTSE and it gets its name due to the fact that the method is easily derived from the complex Taylor series

$$f(x_o + ih) = f(x_o) + f^{(1)}(x_o) \cdot \frac{i \cdot h}{1!} - f^{(2)}(x_o) \cdot \frac{h^2}{2!} - f^{(3)}(x_o) \cdot \frac{i \cdot h^3}{3!} + \ldots \tag{16}$$

Taking the imaginary part of the Taylor series results in

$$imag(f(x_o + ih)) = f^{(1)}(x_o) \cdot \frac{h}{1!} - f^{(3)}(x_o) \cdot \frac{h^3}{3!} \ldots \tag{17}$$

which is solved for $f^{(1)}(x_o)$

$$f^{(1)}(x_o) = \frac{imag(f(x_o + ih)}{h} + f^{(3)}(x_o)\frac{h^2}{3!} \ldots \tag{18}$$

It can be noted that the error due to the approximation is still $O(h^2)$, however there is no difference equation in the first derivative, meaning that there is no lower limit to the selected step size.

CTSE can also be used to determine higher order derivatives. Taking the real part of the Taylor series and rearranging will allow for the calculation of the second derivative.

$$real(f(x_o+ih)) = f(x_o) - f^{(2)}(x_o)\cdot\frac{h^2}{2!} + f^{(4)}(x_o)\cdot\frac{h^4}{4!}\ldots$$
$$\Downarrow \tag{19}$$
$$f^{(2)}(x_o) = \frac{2(f(x_o) - real(f(x_o+ih)))}{h^2} - 2\cdot f^{(4)}(x_o)\cdot\frac{h^2}{4!}\ldots$$

In order to obtain estimates for the higher derivatives, additional sampling points must be taken. The Taylor series of each sample point can be coupled to create a system of equations, that can be solved for the desired derivatives as seen in equation 20.

$$\begin{aligned} imag(f(x_o+ih)) &= f^{(1)}(x_o)\cdot\frac{h}{1!} - f^{(3)}(x_o)\cdot\frac{h^3}{3!} + f^{(5)}(x_o)\cdot\frac{h^5}{5!}\ldots \\ imag(f(x_o+i2h)) &= f^{(1)}(x_o)\cdot\frac{2h}{1!} - f^{(3)}(x_o)\cdot\frac{2^3h^3}{3!} + f^{(5)}(x_o)\cdot\frac{2^5h^5}{5!}\ldots \end{aligned} \tag{20}$$

It is clearly seen from equations 19 and 20 that the calculation of the second order and higher derivatives requires difference operations, and as such the method will face problems similar to those of finite differencing in that h must not be too small nor too big. CTSE does have the advantage that only one sample point is needed to calculate both the first and second derivatives, however. Central differencing requires two sample points to obtain the first two derivatives.

Sampling need not be limited solely to the axes though. Sample points can be selected at any point in the complex plane. For instance in FD sample points are located along a contour encircling the initial point as seen in figure 1.C. FD is a numerical technique based on the Cauchy integral theorem.

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint\frac{f(\xi)}{(\xi - z)^{n+1}}d\xi \tag{21}$$

Derivatives of any order can thus be calculated by evaluating a simple contour integral. One way to evaluate this integral is to take advantage of its similarities to the Fourier integral. Through a change in variables,

$$z \to z_o, \text{ and } \xi = z_o + ce^{i\theta} \tag{22}$$

the Cauchy integral is transformed into a Fourier integral

$$f^{(n)}(z_o) = \frac{n!}{2\pi c^n}\int_0^{2\pi} f(z_o + ce^{i\theta})e^{-in\theta}d\theta. \tag{23}$$

This means that the function *f(z)* can be evaluated by using Fourier transforms. If the function in the Cauchy integral theorem *f(z)* is analytic in a circular neighborhood centered on z_o with a radius of h, then its Taylor series exists and converges within that neighborhood. The resulting Taylor series is

$$f(z_o + ce^{i\theta}) = \sum_{n=0}^{\infty}\frac{f^{(n)}(z_o)}{n!}c^n e^{in\theta} \tag{24}$$

Notice that this function is periodic in θ having period 2π. Also note that each term in the series oscillates at positive integer multiples of θ. Because of this pattern of oscillation, this Taylor series is seen to take on the properties of a periodic Fourier series. In addition each term in the Taylor series is orthogonal to all other terms on the interval *0* to 2π.

$$f(z_o + ce^{i\theta}) \cong \sum_{n=0}^{N} a_n e^{in\theta} \tag{25}$$

Sampling the function $f(z_o+he^{i\theta})$ at several values of θ between *0* and 2π, and then taking the FFT of the sampled data, allows for the calculation of the Fourier coefficients a_n. These coefficients can then be related back to the Taylor series terms, and hence the derivatives of the function.

$$\frac{f^{(n)}(z_o)}{n!} = \frac{a_n}{c^n} \tag{26}$$

The number of derivatives that can be calculated is directly related to the number of sample points chosen. If *N* sample points are selected, then the first *N/2* terms in the Taylor series can be calculated. Due to the orthogonality of the Fourier series terms, the order the estimation error is also related to the number of sample points chosen. For example, when the first 8 terms of the Taylor series are computed, then the estimation error will be $O(h^8)$ for every derivative. This allows for the use of much bigger sampling radii, at least an order of magnitude larger, than the one's employed in central

differencing and CTSE. Unfortunately, error occurs in the FFT when the sampling radius becomes too small, just as with the other two methods.

B. Fatigue Analysis

A bilinear Paris Law relationship was chosen to model the *da/dN* vs. *ΔK* curve. Material properties for the model were taken from experimental data for titanium 6-4. The crack was chosen to be a small, two-dimensional, centered surface crack. Variable amplitude loading was included in the model, and again experimental data was used for the load history. A forward Eulerian differential equation solver was written in Matlab to solve the crack growth equation, as this is the standard solution method in the fatigue field. This differential equation solver returned the number of cycles to failure as well as the final length and depth of the crack.

III. Results

It is assumed in numerical differentiation that the function being differentiated and its derivatives are smooth functions. This is likely not the case for fatigue analysis since the function being differentiated is a first order numerical solution to a differential equation. This lack of smoothness will prevent the accurate calculation of derivatives of an arbitrary order. It may be possible to calculate derivatives up to a certain order if the numerical solution and its first few derivatives are nearly smooth over a local region around the initial point of differentiation. In order to test the validity and performance of the three numerical differentiation techniques, first and second order derivatives were calculated at several points in a small neighborhood. The differential equation solver allows the user to specify the maximum percent growth in crack length or depth per solution step, referred to as the step size of the differential equation solver. Decreasing the step size increases the amount of time required to complete the analysis but it results in smoother and more accurate solutions. Figure 2 displays the first order sensitivity of the number of cycles to failure with respect to the initial crack depth as calculated by CD, with $h = 0.0001$, and figure 3 displays the second order sensitivity calculated by CD for three different step sizes: 2%, 1%, and 0.1%.

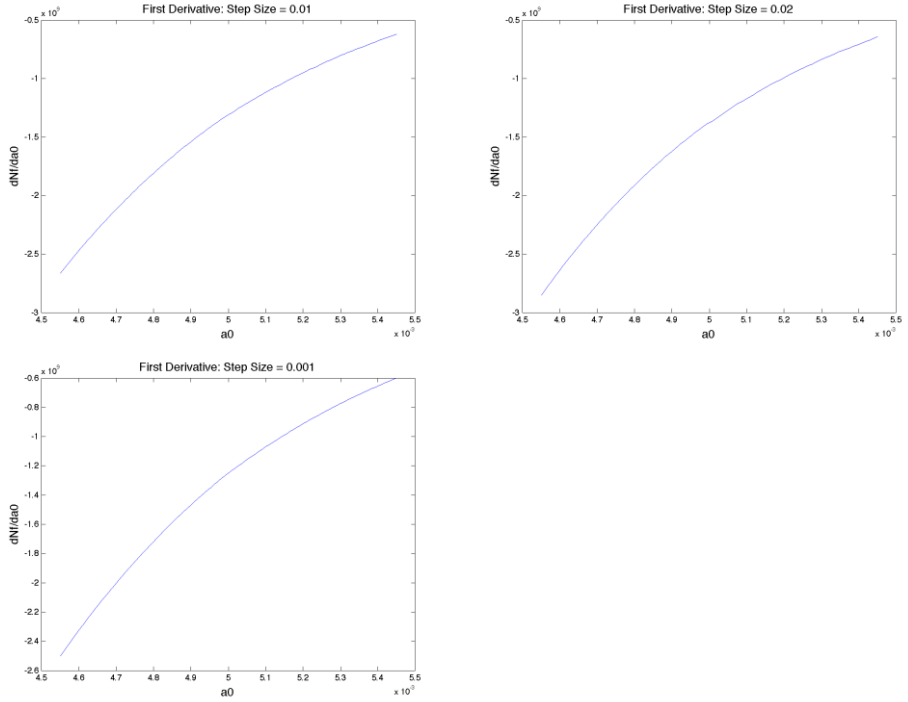

Figure 2. The First Derivative of the Number of Cycles to Failure with Respect to the Initial Crack Depth for 3 Different Step Sizes Calculated by CD

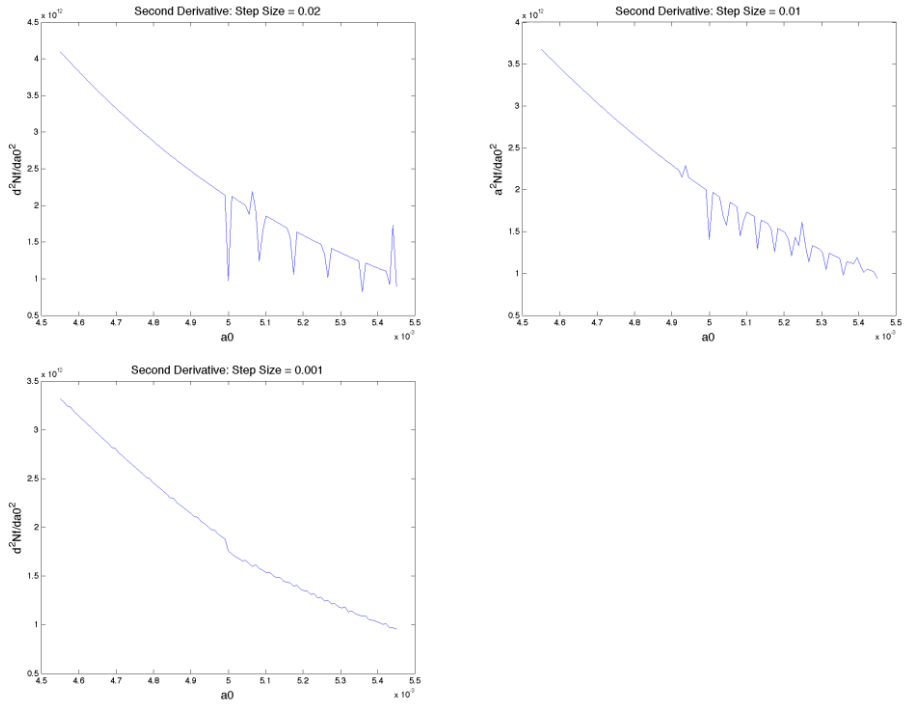

Figure 3. The Second Derivative of the Number of Cycles to Failure with Respect to the Initial Crack Depth for 3 Different Step Sizes Calculated by CD

From these results it is shown that first and second order derivatives can be calculated smoothly across a local range of the input variable. It is also shown that decreasing the step size increases the smoothness of the second derivative and allows for accurate calculation of the derivatives. The first and second order sensitivities calculated by CTSE with *h=0.0001*, are shown in figures 4 and 5 respectively.

These results show more stability at every differential equation step size than the derivatives calculated via CD for the second derivative, while maintaining stable first derivatives. Derivatives were also calculated using a four point FD method with the sampling radius *c=0.0001*. For the four point FD method the operations performed by the FFT result in derivatives that are identical to the average of the CTSE and CD results. As such the behavior of the sensitivities over a local neighborhood is similar to that of CD. The results for FD are not pictured. Sensitivities were also calculated with respect to several other parameters such as material properties like the Paris constant, and the maximum applied stress. In all of these cases CTSE returned smoother first and second order derivatives.

The real advantage of FD over CTSE and CD is that it allows for the accurate calculation of higher order derivatives if they are available, i.e. smooth. Using an 8 point FD method with *c=0.0003*, the first 6 sensitivities of the number of cycles to failure were calculated with respect to the initial crack length. These results are shown for a differential equation step size of 1% in figure 6 and .1% in figure 7.

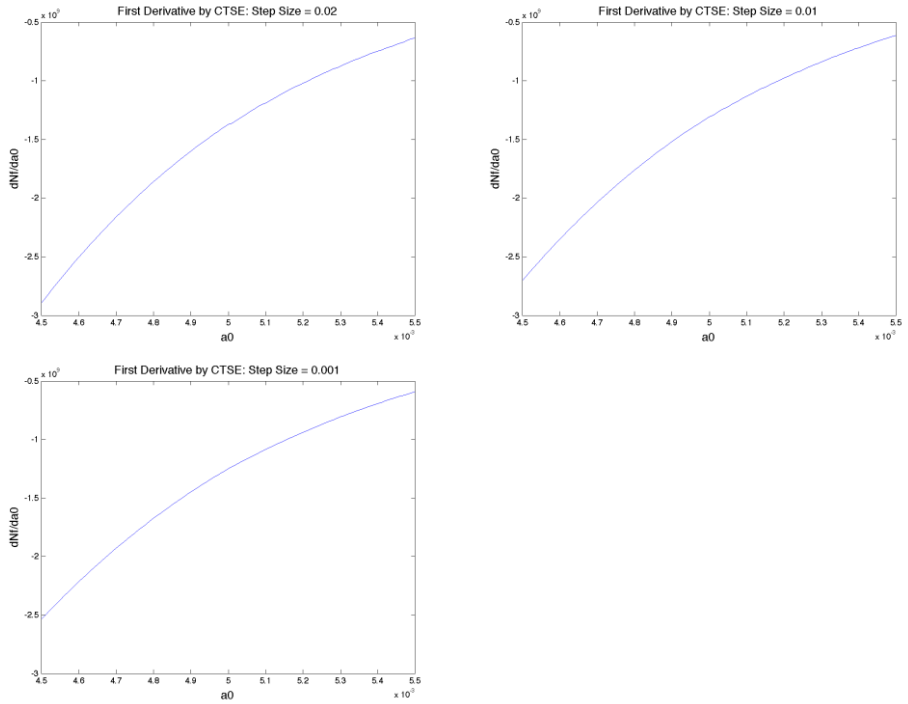

Figure 4. The First Derivative of the Number of Cycles to Failure with Respect to the Initial Crack Depth for 3 Different Step Sizes Calculated by CTSE

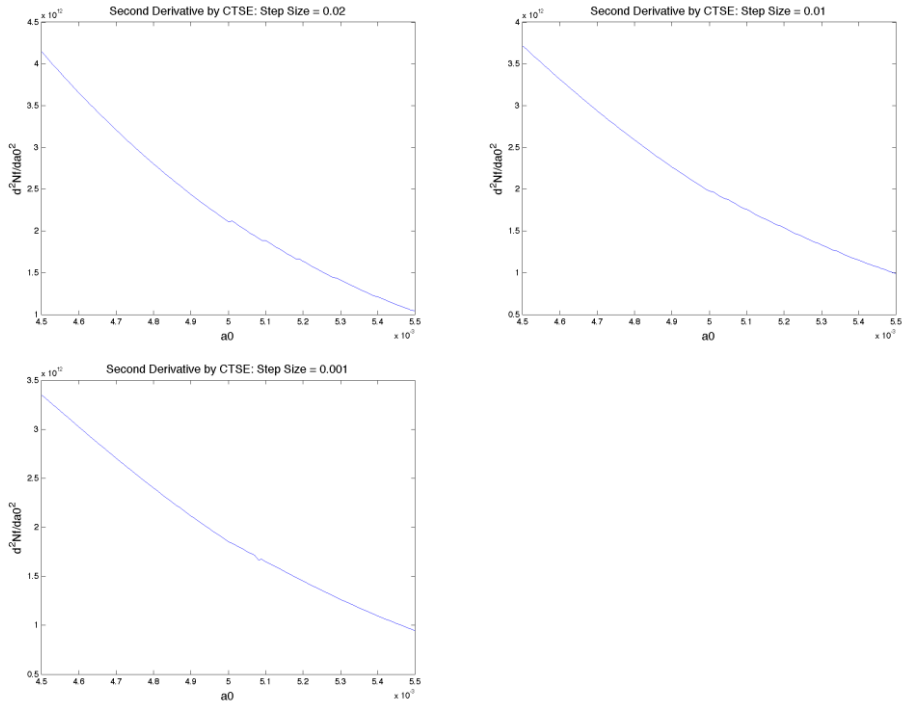

Figure 5. The Second Derivative of the Number of Cycles to Failure with Respect to the Initial Crack Depth for 3 Different Step Sizes Calculated by CTSE

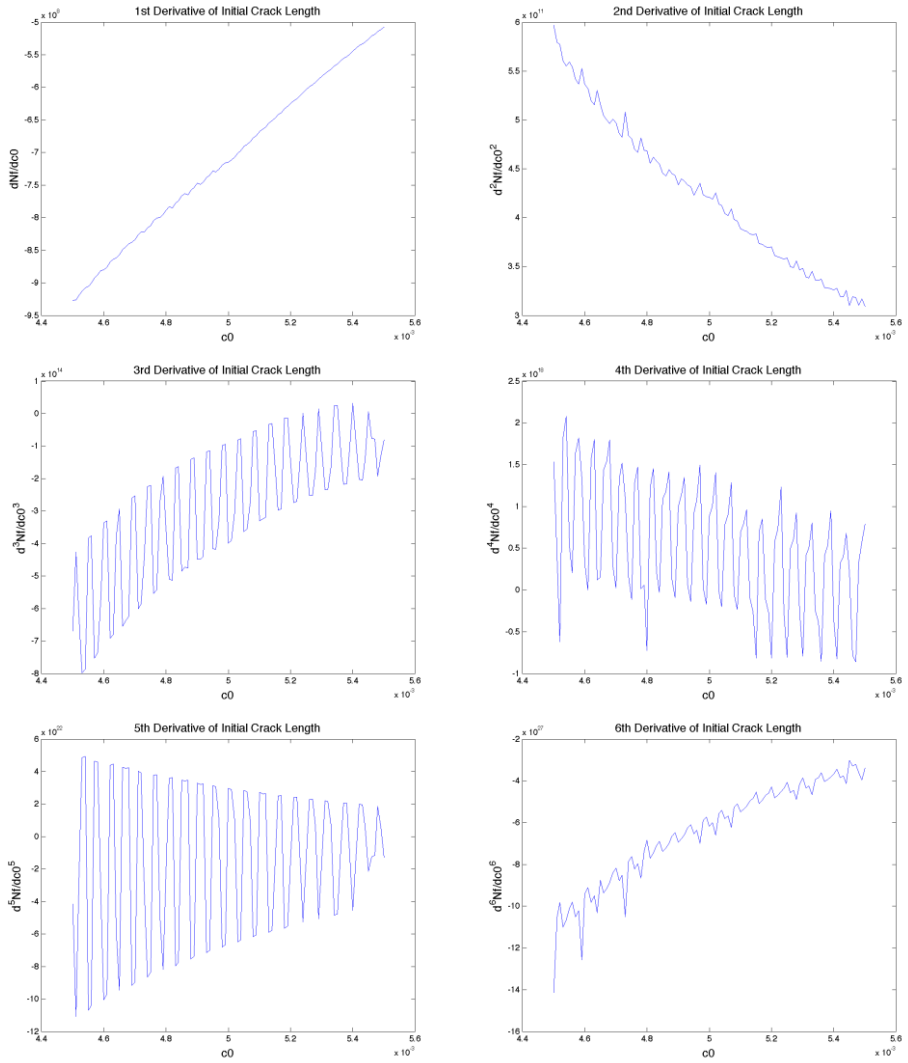

Figure 6. The First Six Derivatives of the Number of Cycles to Failure With Respect to the Initial Crack Length as Calculated by 8 point FD. For a Differential Equation Solver Step Size of 1%

It is seen that for differential equation solver step size of 1% that only the first order sensitivity displays good stability. However reducing the step size dramatically increases the smoothness of the higher order derivatives. With enough computational effort it may be possible to calculate the higher order derivatives with good stability over the local domain. It is also important to note that the sensitivities with respect to the initial crack length show the least stability. Although not shown, sensitivities with respect to the Paris constants showed excellent stability through the sixth derivative at a differential equation solver step size of 1%.

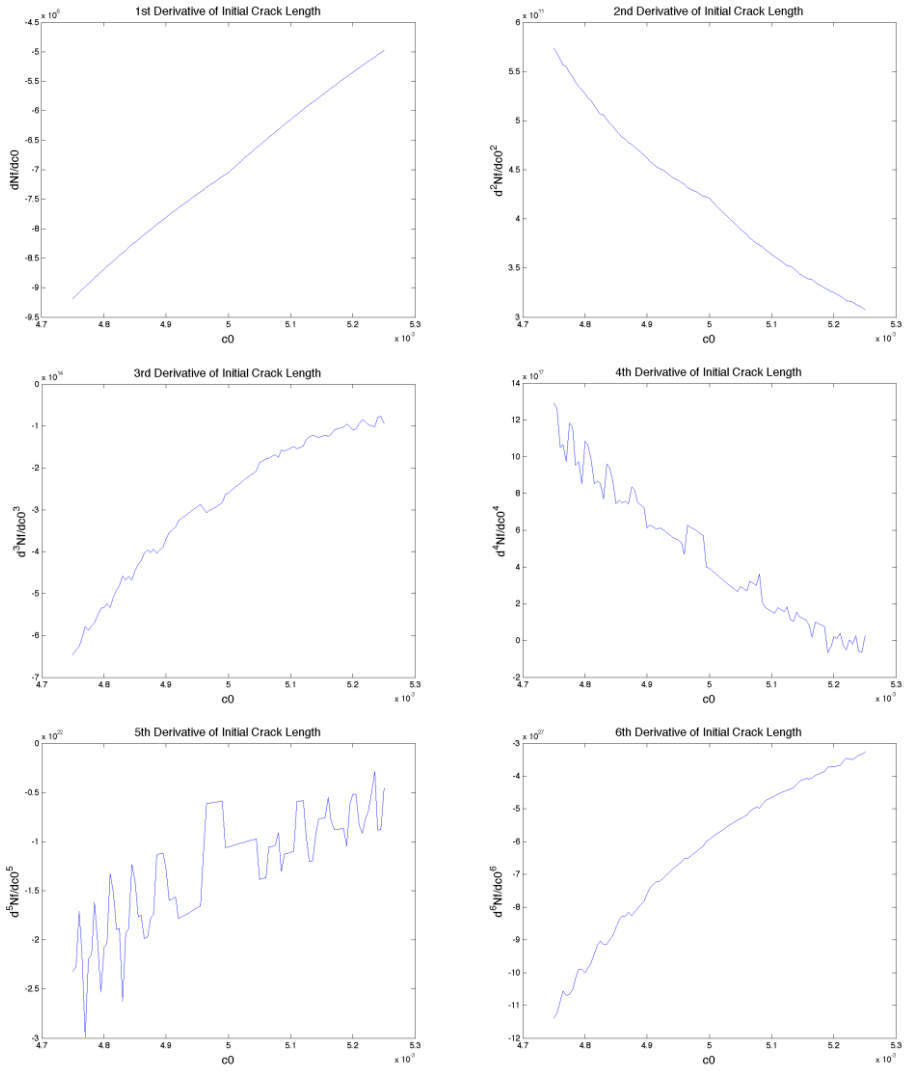

Figure 7. The First Six Derivatives of the Number of Cycles to Failure With Respect to the Initial Crack Length as Calculated by 8 point FD. For a Differential Equation Solver Step Size of 0.1%

IV. Discussion and Conclusions

Complex variable methods numerical differentiation techniques can be used to calculate sensitivities useful for fatigue analysis. CTSE outperforms the traditional CD method for the calculation of first and second order sensitivities. In addition to being more stable across a local domain including the initial point, the CTSE method also requires fewer sampling points, which may increase calculation speed depending on the fatigue analysis code.

All numerical differentiation methods showed more stability in the derivatives when the step size of the differential equation solver was decreased. Decreasing the step size increases the smoothness of the function and its derivatives, which allows for the more accurate calculation of derivatives. This result also hints that the use of more accurate differential equation solver such as a fourth order Runge-Kutta scheme may increase the accuracy and stability of the numerical differentiation techniques.

FD does not outperform CTSE for the calculation of low-order sensitivities but it is a useful tool in the calculation of higher order derivatives. FD can be used to calculate smooth sixth order sensitivities for some input variables. For cases where the high order sensitivities are not very stable, decreasing the step size of the differential equation solver increases the stability. Thus given enough computational power, high order derivatives for any input variable can be calculated via FD.

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